Analysis for a Class of Winner-Take-All Model

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***Abstract*— Recently we have proposed a simple circuit of winner-take-all (WTA) neural network. Assuming no external input, we have derived an analytic equation for its network response time. In this paper, we further analyze the network response time for a class of winner-take-all circuits involving self-decay and show that the network response time of such a class of WTA is the same as that of the simple WTA model.**

***Index Terms*— Inputless winner-take-all neural network, network response time, self-decay.**

I. Introduction

The winner-take-all (WTA) network has been playing a very important role in the design of most of the design of the unsupervised learning neural networks [2], such as competitive learning and Hamming networks. To realize a WTA model, various methods have recently been proposed. Lippman proposed a discrete-time algorithm called Maxnet in order to realize the Hamming network [3]. Majani *et al.* [4] and Dempsey and McVey [5] proposed models based on the Hopfield network topology [6]. Lazzaro *et al.* [7] designed and fabricated a series of compact CMOS integrated circuits for realizing the WTA function. Recently, Seiler and Nossek [8] have proposed an inputless WTA cellular neural-network-based on Chua's CNN [9]. In order to improve on the robustness of this CNN type WTA, Andrew [10] extended Seiler-Nossek model by introducing a clipped total feedback.

Except maxnet, the dynamical equations for most of the above models are governed by many parameters. Therefore, the design and analysis of such networks are complicated. To alleviate such design difficulty, we have recently proposed in [1] a simple analog circuit for WTA with its dynamical equation being governed by just one parameter. It not just simplifies the task for designing the network, but also makes the analysis on the network response time become feasible. In [1], an analytic equation for the response time of such a WTA circuit has been derived and confirmed by intensive computer simulation.

As we have mentioned that WTA is an important component in many unsupervised learning models, the information on its response time is important for investigating. Yet, only a few publications have appeared to provide in-depth analysis on the network response time. In this paper, we apply the same technique to analyze the network reponse time of a class of WTA network which involves self-decay. We will show that the network response time of the decay type WTA is indeed the same as the nondecay type WTA.

The rest of this paper is organized as follows. The next section will introduce the simple and the general self-decay type WTA model. Certain properties governing the derivation of the analytic equation for the network response time will be stated in Section III. Section IV reviews the network response time of the nondecay model. In Section V, the network response time for the self-decay type WTA model will be derived. Comparing with the network reponse times of both models, it will be found that the network response time for the simple nondecay WTA is actually identical to the one for the self-decay type. In order to confirm that the analytical equation can closely approximate the actual network response time, intensive computer simulations have been carried out for the self-decay type model. The result will be reported in Section VI. Using the results obtained in Sections V and VI, three simple methods for designing the WTA model will be presented in Section VII. Finally, a conclusion will be presented in Section VIII.

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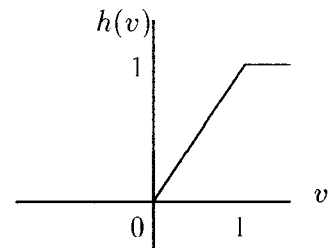

Fig. 1. The input–output characteristic of h.

II. Network Model

We consider an N-neurons fully connected inputless WTA neural network. For the ith neuron, $i = 1, \cdots, N$, the state potential (state variable) and the output of the neuron are denoted by $v_i(t)$ and h_i, respectively, for simplicity, we assume that h_i is a piecewise linear function of v_i as shown in Fig. 1

$$h_i = h(v_i) = \begin{cases} 1, & \text{if } v_i > 1 \\ v_i, & \text{if } 0 \le v_i \le 1 \\ 0, & \text{if } v_i < 0. \end{cases} \quad (1)$$

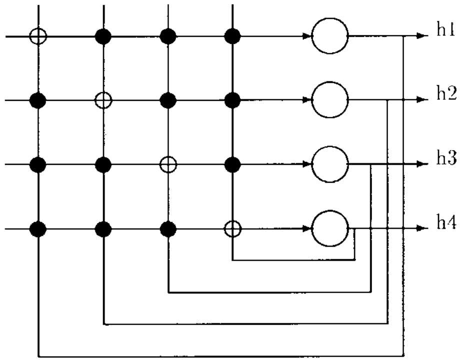

Fig. 2. The network architecture of the proposed WTA neural network. The hollow circles correspond to excitatory connections while the solid black circles correspond to inhibitory connections.

A. Simple WTA Model

In our proposed model [1], the output of each neuron is connected to all the other neurons and itself, in the same way as *Maxnet*. The connection is excitatory if the output is self-fedback. It is inhibitory when the connection is interneuron. The network dynamics can be described as follows:

$$\frac{dv_i(t)}{dt} = h(v_i(t)) - \epsilon \sum_{k=1}^{N} h(v_k(t)) \tag{2}$$

for all $i = 1, \cdots, N$ and $\frac{1}{2} < \epsilon < 1$. Fig. 2 shows the structure of this simple model. The condition on ϵ is used to assure that $(dv_i/dt) < 0$ if the ith neuron is not the winning neuron for all time and $(dv_{\pi_N}/dt) > 0$ when the output of the nonwinning neurons have reached zero.

B. General Model

For some models such as the one described by Seiler-Nossek [8], a decay term $-v_i(t)$ is usually involved in the dynamical equation

$$\frac{dv_i(t)}{dt} = -\beta v_i(t) + h(v_i(t)) - \epsilon \sum_{k=1}^{N} h(v_k(t)) \tag{3}$$

where $0 < \beta$. In this case, even for the winner, the state potential will also decay to zero as $t \to \infty$ and β is too large. This general WTA model has been proposed for a long time. However, the bound on its response time has not been studied.

III. Properties

To simplify the discussion, it is assumed that the initial state potentials can be arranged in a strictly ascending order, i.e., $v_{\pi_1}(0) < v_{\pi_2}(0) < \cdots < v_{\pi_N}(0)$, for a suitable index set $\{\pi_1, \cdots, \pi_N\}$. Now, let us present some properties of the simple WTA model (2) which are useful for the later discussion.

Lemma 1: $\forall\, i, j = 1, \cdots, N$, if $v_i(0) < v_j(0)$, then

1) $v_i(t) < v_j(t)$;
2) $(dv_i/dt) \le (dv_j/dt)$, and equality holds when both $h_i(v_i), h_j(v_j) \in \{0, 1\}$; and
3) $(dv_i/dt) < 0$ if $i \neq \pi_N$.

Proof: From (2)

$$\frac{d}{dt}[v_j(t) - v_i(t)] = h_j(v_j) - h_i(v_i).$$

As v_i and v_j can be located in one of the three regions: $(v_{\min}, 0], (0, 1)$, and $[1, v_{\max})$, there are six cases to be considered

$$\frac{d}{dt}[v_j(t) - v_i(t)] \begin{cases} = 0, & \text{if } 1 < v_i < v_j \\ > 0, & \text{if } 0 < v_i < 1 < v_j \\ > 0, & \text{if } 0 < v_i < v_j < 1 \\ > 0, & \text{if } v_i < 0 < v_j < 1 \\ = 0, & \text{if } v_i < v_j < 0 \\ = 0, & \text{if } v_i < 0, 1 < v_j. \end{cases} \tag{4}$$

That is to say, $(d/dt)[v_j(t) - v_i(t)]$ is nonnegative. Therefore, it is obvious that $v_j(t) - v_i(t) \ge v_j(0) - v_i(0)$ for all $t > 0$ if $v_j(0) - v_i(0) > 0$ and $(dv_i/dt) \le (dv_j/dt)$. Equality holds when $v_j > v_i > 1, v_i < v_j < 0$ or $v_i < 0, v_j > 1$. In other words, it corresponds to the case that $h_i, h_j \in \{0, 1\}$. The proof of Lemma 1(c) can be accomplished by substituting $h_i(v_i) \le h_{\pi_N}(v_{\pi_N})$ into (2) and noting that $\epsilon > 0.5$. □

Theorem 1: If

$$v_{\pi_1}(0) < v_{\pi_2}(0) < \cdots < v_{\pi_N}(0)$$

then

$$v_{\pi_1}(t) < v_{\pi_2}(t) < \cdots < v_{\pi_N}(t)$$

for all $t > 0$.

Proof: The proof is directly implied from Lemma 1(a). □

Theorem 1 and Lemma 1 imply that the time for v_{π_1} reaching zero is finite.

Theorem 2: If $v_{\pi_1}(0) < v_{\pi_2}(0) < \cdots < v_{\pi_N}(0)$, then there exists $T_1 < \infty$, such that $0 = v_{\pi_1}(T_1) < v_{\pi_2}(T_1) < \cdots < v_{\pi_N}(T_1)$.

Proof: Since dv_i/dt is strictly negative for all $i \neq \pi_N, v_{\pi_i}(t)$ is a strictly monotonically decreasing function with regard to time t. So, there exists $T_1 < \infty$ such that $v_{\pi_1}(T_1) = 0$. The proof is completed with the results from Theorem 1. □

Instead of considering the dynamics of the state, $v_i(t)$, we can consider the output dynamics. It aids to the later discussions on the network response time. Since

$$\frac{dh_i}{dt} = \frac{dh_i}{dv_i}\frac{dv_i}{dt} \tag{5}$$

whenever $0 < v_i < 1$, we can express dh_i/dt in terms of $h_1, h_2, \cdots, h_N$, i.e.,

$$\frac{dh_i(t)}{dt} = \begin{cases} 0, & \text{if } h_i(t) = 1 \\ h_i(t) - \epsilon \sum_{k=1}^{n} h_k(t), & \text{if } 0 < h_i(t) < 1 \\ 0, & \text{if } h_i(t) = 0. \end{cases} \tag{6}$$

Using (6) and Lemma 1, the following Lemma and Theorem are deduced.

Lemma 2: For all $i, j = 1, \cdots, N$, if $v_i(0) < v_i(0)$, then

1) $h_i(t) \le h_j(t)$, equality holds when $h_i, h_j \in \{0, 1\}$; and
2) $(dh_i(t)/dt) \le 0$ if $i \neq \pi_N$.

Proof: As $v_i(0) < v_j(0) \rightarrow v_i(t) < v_j(t), v_i(0) < v_j(0)$ implies that $h_j(t) - h_i(t) \geq 0$. Equality holds when $1 \leq v_i(t) \leq v_j(t)$ or $v_i(t) < v_j(t) \leq 0$, i.e., $h_i(t), h_j(t) \in \{0, 1\}$. Proof of Lemma 2(a) is completed. Proof of Lemma 2(b) is similar to the proof of Lemma 1. Since $(dh_i/dt) = (dh_i/dv_i)(dv_i/dt), (dh_i(t)/dt) \leq 0$. The equality holds when $(dh_i(t)/dt) = 0$, i.e., $h_i(t), h_j(t) \in \{0, 1\}$. Then the proof of Lemma 2(b) is completed. □

Theorem 3: If $v_{\pi_1}(0) < v_{\pi_2}(0) < \cdots < v_{\pi_N}(0)$, then there exists $T_1 < \infty$, such that $0^+ = h_{\pi_1}(T_1) < h_{\pi_2}(T_1) < \cdots < h_{\pi_N}(T_1)$.

Proof: The proof is directly implied from Theorem 2. □

Noted that Theorem 1 and Lemma 1 hold true once $t \geq 0$. Although Theorem 3 holds true only when $0 \leq t \leq T_1 < \infty$, it can be generalized to any $\pi_i \neq \pi_1$ or π_N. It is especially important in the discussion of the response time of the network.

So far, we have not analyzed whether the π_N neuron will reach one earlier than the π_1 neuron reaching zero. At the end of the next section, we will show that it is not assured. For instance, when $N = 3, v_1 = 0.1392, v_2 = 0.4503$, and $v_3 = 0.9894$, in which the 2nd neuron will be the last one settling down. We have tried 100 tests; the initial states were initialized randomly. It is found that there are only seven exceptions including the one mentioned. For the rest of the 93 cases, π_1 and π_2 will reach zero first and then π_3 reaches one last. For all of the exceptional cases, the response time is less than three time units. In order to simplify discussion, we assume the following.

Assumption 1: The winner neuron π_N is the last one settling.

Theorem 4: If $0 < v_{\pi_1}(0) < v_{\pi_2}(0) < \cdots < v_{\pi_N}(0) < 1$, then there exists $0 < T_1 < T_2 < \cdots < T_{N-1} < \infty$ such that for

$$h_{\pi_i}(t) = 0 \quad \forall t \geq T_i.$$

Proof: According to Lemma 1(3), $(dv_i/dt) < 0$, there exist $T_2, \cdots, T_{N-1}$ such that

$$v_{\pi_1}(T_2) < v_{\pi_2}(T_2) = 0 < \cdots < v_{\pi_{N-1}}(T_2) < v_{\pi_N}(T_2)$$
$$\cdots$$
$$v_{\pi_1}(T_{N-1}) < \cdots < v_{\pi_{N-1}}(T_{N-1}) = 0 < v_{\pi_N}(T_{N-1}).$$

Based on the definition of $h_{\pi_i}(v_{\pi_i})$, the above implies that there exists $T_1 < T_2 < \cdots < T_{N-1} < \infty$ such that $h_{\pi_i}(t) = 0 \quad \forall t \geq T_i$. □

In the sequel, we can deduce that the response time of the network is finite.

Theorem 5: If $0 < v_{\pi_1}(0) < v_{\pi_2}(0) < \cdots < v_{\pi_N}(0) < 1$, then there exists $T_N < \infty$ such that $\forall\, t > T_N$

$$h_i(t) = \begin{cases} 1, & \text{if } i = \pi_N \\ 0, & \text{if } i \neq \pi_N \end{cases}$$

where $i = 1, 2, \cdots, N$.

Proof: According to Theorem 4, $h_{\pi_1} = h_{\pi_2} = \cdots = h_{\pi_{N-1}} = 0$, when $t \geq T_{N-1}$. From Lemma 2(a), we deduce that $h_{\pi_N} > h_{\pi_{N-1}} = 0$. This brings out the following two cases to be considered: 1) $h_{\pi_N}(T_{N-1}) = 1$ and 2) $h_{\pi_N}(T_{N-1}) < 1$. In the former case, $T_N < \infty$ since $T_N \leq T_{N-1}$. In the latter case, h_{π_N} needs time to rise to one. Since the output dynamics of h_{π_N} is governed by

$$\frac{d}{dt} h_{\pi_N}(t - T_{N-1}) = (1 - \epsilon) h_{\pi_N}(t - T_{N-1}) \tag{7}$$

for all $t \geq T_{N-1}$. So

$$h_{\pi_N}(t - T_{N-1}) = h_{\pi_N}(T_{N-1}) e^{(1-\epsilon)(t - T_{N-1})}$$

for all $t \geq T_{N-1}$. When $h_{\pi_N}(t - T_{N-1}) = 1$

$$T_N = T_{N-1} - \frac{1}{1-\epsilon} \log\left(h_{\pi_N}(T_{N-1})\right) < \infty.$$

The inequality holds true since $0 < h_N(T_{N-1}) < 1$. Therefore, $T_N < \infty$. Hence the proof is completed. □

Theorem 5 shows clearly that our proposed network can function as a WTA neural network and its response time is finite. The only restriction is that $v_i(0) \neq v_j(0)$ if $i \neq j$. It is worthy noting that this condition is a far more relaxed one than that of the design conditions derived in the Seiler-Nossek model [8]. Besides, our analysis does not depend on the number of neurons in the network. Consequently, the optimal design of our WTA network as well as the analysis of the network response time can be made relatively simple.

It should be noted that Theorems 1–4 hold true for the general model as well. However, Theorem 5 holds true only when $\beta + \epsilon < 1$. In case $\beta + \epsilon > 1, h_{\pi_N}$ will decay to zero. This property for the general model can be stated in the following Theorem.

Theorem 6: If $v_{\pi_1}(0) < v_{\pi_2}(0) < \cdots < v_{\pi_N}(0)$, then there exists $T_N < \infty$ such that $\forall t > T_N$

$$h_i(t) = 0$$

where $i = 1, 2, \cdots, N-1$ and h_{π_N} will satisfy one of the following conditions: 1) If $\beta + \epsilon > 1, \lim_{t\to\infty} h_{\pi_N}(t) = 0$; 2) If $\beta + \epsilon = 1, \lim_{t\to\infty}$ will converge to a constant value between zero and one; and 3) If $\beta + \epsilon < 1, \lim_{t\to\infty} h_{\pi_N}(t) = 1$.

Proof: Follow Theorem 4, at time $T_{N-1}, h_{\pi_i}(T_{N-1}) = 0$ for all $i = 1, \cdots, N-1$, and

$$\frac{dv_N(t)}{dt} = -\beta v_N(t) + (1 - \epsilon) h(v_N(t)). \tag{8}$$

As $0 < h_{\pi_N}(t) < 1, h_{\pi_N}(t) = v_{\pi_N}(t)$ and

$$\frac{dh_{\pi_N}(t)}{dt} = -\beta h_{\pi_N}(t) + (1 - \epsilon) h_{\pi_N}(t).$$

Obviously, when $\beta + \epsilon > 1, (dh_{\pi_N}(t)/dt) < 0$. Hence the output of the winner node will decrease to zero. Similarly, when $\beta + \epsilon > 1, (dh_{\pi_N}(t)/dt) > 0$, the output of the winner node will rise to one. when $\beta + \epsilon = 1, (dh_{\pi_N}(t)/dt) = 0, h_{\pi_N}(t) = h_{\pi_N}(T_{N-1})$ for all time $t \geq T_{N-1}$. Then the proof is completed. □

IV. Network Response Time of the Simple WTA Model

We can proceed to see what will happen immediately after T_1. Once $t \geq T_1$ and from Theorems 2–4

$$h_{\pi_1}(t) = 0, \quad \frac{dh_{\pi_1}(t)}{dt} = 0$$

and

$$\begin{bmatrix} \dot{h}_{\pi_2}(t) \\ \dot{h}_{\pi_3}(t) \\ \cdots \\ \dot{h}_{\pi_N}(t) \end{bmatrix} = \begin{bmatrix} 1-\epsilon & -\epsilon & \cdots & -\epsilon \\ -\epsilon & 1-\epsilon & \cdots & -\epsilon \\ \cdots & \cdots & \cdots & \cdots \\ -\epsilon & -\epsilon & \cdots & 1-\epsilon \end{bmatrix} \begin{bmatrix} h_{\pi_2}(t) \\ h_{\pi_3}(t) \\ \cdots \\ h_{\pi_N}(t) \end{bmatrix}.$$

Obviously, the output dynamic is now governed by an $(N-1)$-dimensional first-order differential equation. Let us denote

$$\hat{h}_N(t) = (h_{\pi_1}(t), h_{\pi_2}(t), \cdots, h_{\pi_N}(t))'$$

for all $0 < t < T_1$ and

$$\hat{h}_{N-1}(t) = (h_{\pi_2}(t), \cdots, h_{\pi_N}(t))'$$

when t is just greater than T_1, where $'$ denotes transpose. Note that π_N is the index of the neuron for which the initial state potential is the largest. When $0 < t < T_1$, we get that

$$\frac{d}{dt}\hat{h}_N(t) = A_N \hat{h}_N(t) \tag{9}$$

and when t is just greater than T_1, we can deduce that

$$\frac{d}{dt}\hat{h}_{N-1}(t) = A_{N-1}\hat{h}_{N-1}(t) \tag{10}$$

where

$$A_k = \begin{bmatrix} 1-\epsilon & -\epsilon & \cdots & -\epsilon \\ -\epsilon & 1-\epsilon & \cdots & -\epsilon \\ \cdots & \cdots & \cdots & \cdots \\ -\epsilon & -\epsilon & \cdots & 1-\epsilon \end{bmatrix}_{k \times k}$$

for $k = N-1, N$. Just after $t = T_1$, the network dynamical equation is changed from (9)–(10). It indicates that system (2) is a reduced-dimension system. Hence T_1 can be evaluated using the following Lemma.

Lemma 3: The eigenvalues of A_N are $(1 - N\epsilon)$ and one. The corresponding eigensubspace of $(1 - N\epsilon)$ and one are M_N and $M_N^{\perp}$, respectively, where

$$M_N = \text{span}\left\{ \underline{e}_{1N} = \left(\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \cdots, \frac{1}{\sqrt{N}} \right)^T \right\}$$

and

$$M_N^{\perp} = \{ \underline{v} \in R^N | \underline{v}^T \underline{e}_{1N} = 0 \}.$$

Proof: See Appendix. □

For all $t \in \{ s \geq 0 | 0 < h_{\pi_i}(s) < 1, i = 1, 2 \cdots, N \}$

$$h_{\pi_i}(t) = e^{(1-N\epsilon)t} \left[\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N} \right] + e^t \left[v_{\pi_i}(0) - \frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N} \right] \tag{11}$$

for all $i = 1, 2, \cdots, N$. Obviously, the output of the π_1 neuron will be the first one reaching zero since $h_{\pi_i} < h_{\pi_j}$ if $i < j$. Hence, T_1 can be evaluated by setting $h_{\pi_1}(t) = 0$

$$T_1 = -\frac{1}{N\epsilon} \log \left[\frac{\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N} - v_{\pi_1}(0))}{\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N}} \right]. \tag{12}$$

Substituting T_1 into (11), we can readily show that

$$h_{\pi_i}(T_1) = \left[\frac{\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N} - v_{\pi_1}(0))}{\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N}} \right]^{(-1/N\epsilon)} (v_{\pi_i}(0) - v_{\pi_1}(0))$$

for all $i = 2, 3, \cdots, N$.

Based on the assumption and (11), we can readily deduce that

$$0 < v_{\pi_N}(t) < 1 \tag{13}$$

and

$$v_{\pi_i}(t) - v_{\pi_j}(t) = h_{\pi_i}(t) - h_{\pi_j}(t) \tag{14}$$

$$= e^{t-T_1}(h_{\pi_i}(T_1) - h_{\pi_j}(T_1)) \tag{15}$$

for all $i, j = 2, 3, \cdots, N$ and $v_{\pi_2}(t) \geq 0$. Similarly, we can evaluate the difference $(h_{\pi_i}(T_1) - h_{\pi_j}(T_1))$ using the same idea and obtain that

$$v_{\pi_i}(t) - v_{\pi_j}(t) = e^t(v_{\pi_i}(0) - v_{\pi_j}(0)) \tag{16}$$

as long as v_{π_i} and v_{π_j} are greater than zero.

Now, consider the time t just after T_1, it is readily deduced that

$$h_{\pi_i}(t) = e^{(1-(N-1)\epsilon)t} \left[\frac{\sum_{k=2}^{N} v_{\pi_k}(0)}{N-1} \right] + e^t \left[v_{\pi_i}(0) - \frac{\sum_{k=2}^{N} v_{\pi_k}(0)}{N-1} \right]. \tag{17}$$

Setting $h_{\pi_2}(t) = 0$, we can deduce T_2 as follows:

$$T_2 - T_1 = -\frac{1}{(N-1)\epsilon} \log \left[\frac{\frac{\sum_{k=2}^{N} v_{\pi_k}(T_1)}{N-1} - v_{\pi_2}(T_1))}{\frac{\sum_{k=2}^{N} v_{\pi_k}(T_1)}{N-1}} \right].$$

Since $h_{\pi_1}(T_1) = v_{\pi_1}(T_1) = 0$, the above equation becomes

$$T_2 - T_1 = -\frac{1}{(N-1)\epsilon} \log \left[\frac{\sum_{k=2}^{N} (v_{\pi_k}(T_1) - v_{\pi_2}(T_1))}{\sum_{k=2}^{N} (v_{\pi_k}(T_1) - v_{\pi_1}(T_1))} \right].$$

Therefore, using the result obtained in (16), we obtain that

$$T_2 - T_1 = -\frac{1}{(N-1)\epsilon} \log \left[\frac{\sum_{k=2}^{N} (v_{\pi_k}(0) - v_{\pi_2}(0))}{\sum_{k=2}^{N} (v_{\pi_k}(0) - v_{\pi_1}(0))} \right]. \tag{18}$$

Using the same technique, we can obtain T_3 to T_{N-1} recursively

$$T_3 - T_2 = -\frac{1}{(N-2)\epsilon} \log \left[\frac{\sum_{k=3}^{N} (v_{\pi_k}(0) - v_{\pi_3}(0))}{\sum_{k=3}^{N} (v_{\pi_k}(0) - v_{\pi_2}(0))} \right]$$

$$\cdots \tag{19}$$

and

$$T_{N-1} - T_{N-2} = -\frac{1}{2\epsilon} \log \left[\frac{\sum_{k=N-1}^{N} (v_{\pi_k}(0) - v_{\pi_{N-1}}(0))}{\sum_{k=N-1}^{N} (v_{\pi_k}(0) - v_{\pi_{N-2}}(0))} \right]. \tag{20}$$

Denote the network response time T_{rt} and define it as T_{N-1}. Then T_{rt} can be written explicitly as follows:

$$T_{rt} = \sum_{j=2}^{N-1} \frac{1}{j\epsilon} \log \left[\frac{\sum_{k=N+1-j}^{N} (v_{\pi_k}(0) - v_{\pi_{N-j}}(0))}{\sum_{k=N+1-j}^{N} (v_{\pi_k}(0) - v_{\pi_{N+1-j}}(0))} \right] + \frac{1}{N\epsilon} \log \left[\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{\sum_{k=1}^{N} (v_{\pi_k}(0) - v_{\pi_1}(0))} \right]. \tag{21}$$

It is interesting to note that the network response time is dependent solely on ϵ and the initial conditions of the neurons only.

V. Network Response Time of the General WTA Model

Consider the case when $\beta > 0$, we can obtain similar equations as (9) and (10)

$$\frac{d}{dt}\hat{h}_k(t) = B_k \hat{h}_k(t), \tag{22}$$

where

$$B_k = \begin{bmatrix} 1-\epsilon-\beta & -\epsilon & \cdots & -\epsilon \\ -\epsilon & 1-\epsilon-\beta & \cdots & -\epsilon \\ \cdots & \cdots & \cdots & \cdots \\ -\epsilon & -\epsilon & \cdots & 1-\epsilon-\beta \end{bmatrix}_{k\times k} = A_k - \beta I_{k\times k}. \tag{23}$$

for $k = 1, 2, \cdots, N-1, N$. Using the results obtained in (9) for A_N, the eigenvalues of B_N can be stated as follows:

$$\lambda_1 = 1 - N\epsilon - \beta \tag{24}$$

$$\lambda_2 = 1 - \beta. \tag{25}$$

Thus, similar to that of (11), we can have an equation for $h_{\pi_i}(t)$ as follows:

$$h_{\pi_i}(t) = e^{(1-N\epsilon-\beta)t} \left[\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N} \right] + e^{(1-\beta)t} \left[v_{\pi_i}(0) - \frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N} \right] \tag{26}$$

for all $i = 1, 2, \cdots, N$. Therefore, the settling time for π_1 is given as follows:

$$T_1 = -\frac{1}{N\epsilon} \log \left[\frac{\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N} - v_{\pi_1}(0))}{\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{N}} \right]. \tag{27}$$

Following the same steps as above, the network response time can be obtained and represented as follows:

$$T_{rtg} = \sum_{j=2}^{N-1} \frac{1}{j\epsilon} \log \left[\frac{\sum_{k=N+1-j}^{N} (v_{\pi_k}(0) - v_{\pi_{N-j}}(0))}{\sum_{k=N+1-j}^{N} (v_{\pi_k}(0) - v_{\pi_{N+1-j}}(0))} \right] + \frac{1}{N\epsilon} \log \left[\frac{\sum_{k=1}^{N} v_{\pi_k}(0)}{\sum_{k=1}^{N} (v_{\pi_k}(0) - v_{\pi_1}(0))} \right]. \tag{28}$$

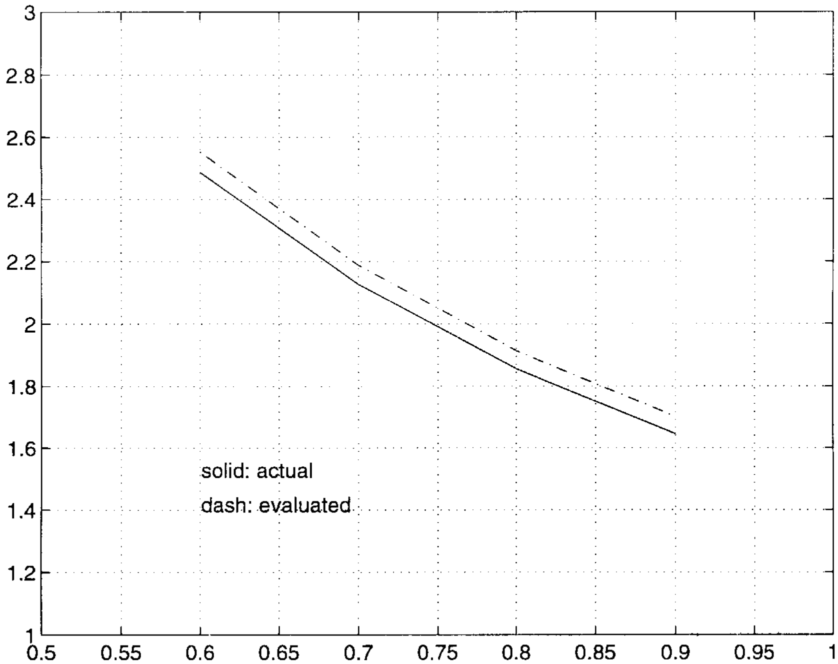

Fig. 3. The average response time of the network for different values of epsilon. The horizontal axis corresponds to the value of epsilon while the vertical axis corresponds to the response time.

Comparing (21) and (28), the network response time for our general WTA model is the same as that of the simple WTA model

$$T_{rtg} = T_{rt}. \tag{29}$$

VI. Simulation Verification

Equation (21) indicates that the network response time relies on two factors: the initial conditions of the neurons' state potentials and the parameter ϵ. But, one may query about the consistency of (21) and the actual network response time because an assumption has been made prior to the derivation of the equation.

In order to demonstrate that the deduced response time can indeed reflect the actual response time, extensive simulations were carried out for different values of ϵ. Four different values of ϵ were examined: 0.6, 0.7, 0.8, and 0.9. For each value of ϵ and particular size (N) of the WTA, 25 sets of simulation were carried out. The size N varied from 4, 8–100. In each set of simulation, 100 runs of the experiment with different initial states (randomly chosen with a uniform probability density function) were carried out.

It is found that when the size of the WTA neural network is small, both the evaluated and experimental values of the settling time are short. As the size of the WTA neural network increases from $N = 20$ to $N = 100$, both the evaluated and experimental values of the settling time manifest trends of steady increase. However, the rate of increase is very small. If we take the average values of the response time for the sizes from $N = 20$ to $N = 100$ and compare the decreasing trend with respect to the value of ϵ, an interesting observation is noted: as shown in Fig. 3, both trends of decreasing suggest an exponential decay. Therefore, using the results shown in Fig. 3, we can design a network with appropriate component values.

VII. Design Examples

As mentioned in the introductory section and the discussion on Section III, the inclusion of the self-decay can provide a flexibility in the design of a WTA. For example, if we want the output of the winner node to decay to zero, we can set $\epsilon + \beta > 1$. Here we give three examples showing how to design the values of ϵ and β based on the results obtained above.

Example 1: Suppose we want to have $\lim_{t\to\infty} h_{\pi_N}(t) = 0$ and the network response time is about two time units. Referring to Fig. 3, we can set ϵ to be 0.5. Then, we set $\beta = 0.6$ to make sure that the output of the winner node will decay to zero.

Example 2: Suppose we want to have $\lim_{t\to\infty} h_{\pi_N}(t)$ equal to a constant value in between zero and one. The network response time is about two time units. Again, we set ϵ to be 0.75. As $\epsilon + \beta = 1$ is the condition for that $\lim_{t\to\infty} h_{\pi_N}(t)$ equal to a constant, we set $\beta = 0.25$.

Example 3: Suppose we want to have $\lim_{t\to\infty} h_{\pi_N}(t) = 1$ and the network response time is about two time unit. We set ϵ to be 0.75. To ensure that the output of the winner node reach one, we can set $\beta = 0.15$.

VIII. Conclusion

In summary, we have reviewed and analyzed the properties of a simple WTA model which has been proposed recently. In particular, as analytic equation for its response time (the time when $h_{\pi_{N-1}} = 0$) is presented—(21). Using the same technique, we have derived an analytic equation (28) for the response time of a general class of WTA which has a dynamical equation involving self-decay. Comparing both equations, it is found that the network response time of the simple model can be treated as an upper bound for a more general[1] class of WTA models. Finally, one should note that the results presented in this paper are preliminary. A more general model with nonunity neuron gain, or infinity gain, and nonunity self-feedback synaptic weight is deserved for further research.

Appendix
Proof of Lemma 3

Let $x_i = (1/\sqrt{N})$ for all i, i.e., $\underline{x} \in M_N$. Then

$$\begin{aligned}(A_N\underline{x})_i &= (1-\epsilon)\left(\frac{1}{\sqrt{N}}\right) - (N-1)\epsilon\left(\frac{1}{\sqrt{N}}\right)\\ &= (1-N\epsilon)x_i.\end{aligned} \tag{30}$$

Hence $A_N\underline{x} = \underline{x}$ as $i = 1,\cdots,N$. And $(1-N\epsilon)$ is an eigenvalue for A_N.

Next consider $\underline{w} = \underline{v} - (\underline{v}^T\underline{e}_{1N})\underline{e}_{1N}$, i.e., $\underline{w} \in M_N^{\perp}$,

$$\begin{aligned}(A_N\underline{w})_i &= (1-\epsilon)w_i - \epsilon\sum_{j\neq i} w_j\\ &= \left(v_i - \sum_{j=1}^{N}\frac{v_j}{N}\right) - \epsilon\sum_{j=1}^{N}\left(v_j - \sum_{k=1}^{N}\frac{v_k}{N}\right)\\ &= \left(v_i - \sum_{j=1}^{N}\frac{v_j}{N}\right)\\ &= w_i.\end{aligned} \tag{31}$$

Hence, $A_N\underline{w} = \underline{w}$ and the other eigenvalue is one. If $\underline{w}$ is replaced by any one of the following vectors:

$$\left(\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0, \cdots, 0\right)^T$$
$$\left(\tfrac{1}{\sqrt{2}}, 0, \tfrac{-1}{\sqrt{2}}, \cdots, 0\right)^T$$
$$\cdots$$
$$\left(\tfrac{1}{\sqrt{2}}, 0, 0, \cdots, \tfrac{-1}{\sqrt{2}}\right)^T.$$

it can be concluded that $(1-\epsilon)$ and one are the only eigenvalues of A_N because

$$\dim(M_N) + \dim(M_N^{\perp}) = N.$$

And the proof for Lemma 3 is completed. □

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[1] As one reviewer has pointed out that the model being discussed is not really general. A more general model with nonunity neuron gain and nonunity self-feedback synaptic weight should be analyzed for practical implementation.

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