IT2025 Supplementary Note 04

October 28, 2025

1 Nine Balls One Abnormal

1.1 Lighter Ball

You are given a set of 9 balls. All of them look the same. Eight of them weight 2000 grams and one of them weights 1999 grams. Human is unable to sense this little difference. Now, the only tool you have is a pan balance. Your job is to find out which of them is lighter. Describe in detail, step by step, how do you use the pan balance to find the lighter ball.

Answer: There are two solutions for solving the above problem.

Solution 1: Initially, label the balls with numbers $B1, B2, \cdots, B9$.

```
---------------------------------------------------------------------------
Step 1: WEIGHT B1 and B2. GOTO Step 2.
Step 2: 2.1: IF B1 < B2, B1 is abnormal. STOP! ELSE GOTO Step 2.2.
        2.2: IF B1 > B2, B2 is abnormal. STOP! ELSE GOTO Step 3.1.
Step 3: 3.1: SET N = 3.
        3.2: WEIGHT B1 and BN.
        3.3: IF B1 > BN, BN is abnormal. STOP! ELSE GOTO Step 3.4.
        3.4: N = N+1. GOTO Step 3.2.
---------------------------------------------------------------------------
```

For the above algorithm, the number of WEIGHT for finding the abnormal ball is no more than 8. It happens if the abnormal ball is B_9.

Solution 2: Initially, arbitrarily partition the balls in three groups, say Group A, B and C. Each group has three balls. Label the balls A_1, A_2 and A_3 for the balls in Group A. Label the balls B_1, B_2 and B_3 for the balls in Group B. Label the balls C_1, C_2 and C_3 for the balls in Group C.

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---------------------------------------------------------------------------
Step 1: WEIGHT (A1,A2,A3) and (B1,B2,B3). GOTO Step 2.
Step 2: IF (A1,A2,A3) < (B1,B2,B3), GOTO Step 5.
Step 3: IF (A1,A2,A3) > (B1,B2,B3), GOTO Step 6.
Step 4: IF (A1,A2,A3) = (B1,B2,B3), GOTO Step 7.
Step 5: 5.1: WEIGHT A1 and A2. GOTO Step 5.2.
        5.2: IF A1 < A2, A1 is abnormal. STOP! ELSE GOTO Step 5.3.
        5.3: IF A1 > A2, A2 is abnormal. STOP!
             ELSE A3 is abnormal. STOP!
Step 6: 6.1: WEIGHT B1 and B2. GOTO Step 6.2.
        6.2: IF B1 < B2, B1 is abnormal. STOP! ELSE GOTO Step 6.3.
        6.3: IF B1 > B2, B2 is abnormal. STOP!
             ELSE B3 is abnormal. STOP!
Step 7: 7.1: WEIGHT C1 and C2. GOTO Step 7.2.
        7.2: IF C1 < C2, C1 is abnormal. STOP! ELSE GOTO Step 7.3.
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```
        7.3: IF C1 > C2, C2 is abnormal. STOP!
             ELSE C3 is abnormal. STOP!
---------------------------------------------------------------------------
```

For the above algorithm, the number of WEIGHT for finding the abnormal ball is no more than 2.

1.2 Heavier Ball

You are given a set of 9 balls. All of them look the same. Eight of them weight 2000 grams and one of them weights 2001 grams. Pan balance is the only tool to be used. Describe in detail, step by step, how do you use the pan balance to find the heavier ball.

Answer: Similar to the solutions for above.

1.3 Unknown Abnormality

Again, you are given a set of 9 balls which are looked and sensed the same. Eight of them are 2000 grams. For the abnormal ball, we do not know if it is lighter or heavier. Describe in detail, step by step, how do you use the pan balance to find the abnormal ball.

Answer: There are two solutions for the above problem.

Solution 1: Initially, label the balls with numbers $1, 2, \cdots, 9$.

```
---------------------------------------------------------------------------
Step 1: WEIGHT B1 and B2. GOTO Step 2.
Step 2: 2.1: IF B1 != B2, GOTO Step 2.2.1.
          2.1.1: WEIGHT B1 and B3.
          2.1.2: IF B1 = B3, B2 is abnormal. STOP!.
                 ELSE B1 is abnormal. STOP!
        2.2: IF B1 = B2, GOTO Step 3.1.
Step 3: 3.1: SET N = 3.
        3.2: WEIGHT B1 and BN.
        3.3: IF B1 != BN, BN is abnormal. STOP! ELSE GOTO Step 3.4.
        3.4: N = N+1. GOTO Step 3.2.
---------------------------------------------------------------------------
```

For the above algorithm, the number of WEIGHT for finding the abnormal ball is no more than 8. It happens if the abnormal ball is B_9.

Solution 2: Initially, arbitrarily partition the balls in three groups, say Group A, B and C. Each group has three balls. Label the balls A_1, A_2 and A_3 for the balls in Group A. Label the balls B_1, B_2 and B_3 for the balls in Group B. Label the balls C_1, C_2 and C_3 for the balls in Group C. Figure 1 shows the decision tree for this problem.

```
---------------------------------------------------------------------------
Step 1: 1.1: WEIGHT (A1,A2,A3) and (B1,B2,B3). GOTO Step 1.2.
        1.2: SET (GA > GB) = TRUE, IF (A1,A2,A3) > (B1,B2,B3).
             GOTO Step 2. ELSE GOTO Step 1.3.
        1.3: SET (GA < GB) = TRUE, IF (A1,A2,A3) < (B1,B2,B3).
             GOTO Step 2. ELSE SET (GA = GB) = TRUE. GOTO Step 2.
Step 2: 2.1: WEIGHT (A1,A2,A3) and (C1,C2,C3). GOTO Step 2.2.
        2.2: SET (GA > GC) = TRUE, IF (A1,A2,A3) > (C1,C2,C3).
             GOTO Step 3. ELSE GOTO Step 2.3.
        2.3: SET (GA < GC) = TRUE, IF (A1,A2,A3) < (C1,C2,C3).
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             GOTO Step 3. ELSE SET (GA = GC) = TRUE. GOTO Step 3.

[Note: After Step 2, it comes up with six possible outcomes only.]

Step 3: 3.1: IF (GA < GB) and (GA < GC), GOTO Step 4.
             ELSE GOTO Step 3.2.
        3.2: IF (GA > GB) and (GA = GC), GOTO Step 5.
             ELSE GOTO Step 3.3.
        3.3: IF (GA = GB) and (GA > GC), GOTO Step 6.
             ELSE GOTO Step 3.4.
        3.4: IF (GA > GB) and (GA > GC), GOTO Step 7.
             ELSE GOTO Step 3.5.
        3.5: IF (GA < GB) and (GA = GB), GOTO Step 8.
             ELSE GOTO Step 3.6.
        3.6: IF (GA = GB) and (GA < GC), GOTO Step 9.

[The abnormal ball is lighter.]
Step 4: 4.1: WEIGHT A1 and A2. GOTO Step 4.2.
        4.2: IF A1 < A2, A1 is abnormal. STOP! ELSE GOTO Step 4.3.
        4.3: IF A1 > A2, A2 is abnormal. STOP!
             ELSE A3 is abnormal. STOP!
Step 5: 5.1: WEIGHT B1 and B2. GOTO Step 5.2.
        5.2: IF B1 < B2, B1 is abnormal. STOP! ELSE GOTO Step 5.3.
        5.3: IF B1 > B2, B2 is abnormal. STOP!
             ELSE B3 is abnormal. STOP!
Step 6: 6.1: WEIGHT C1 and C2. GOTO Step 6.2.
        6.2: IF C1 < C2, C1 is abnormal. STOP! ELSE GOTO Step 6.3.
        6.3: IF C1 > C2, C2 is abnormal. STOP!
             ELSE C3 is abnormal. STOP!

[The abnormal ball is heavier.]
Step 7: 7.1: WEIGHT A1 and A2. GOTO Step 7.2.
        7.2: IF A1 > A2, A1 is abnormal. STOP! ELSE GOTO Step 7.3.
        7.3: IF A1 < A2, A2 is abnormal. STOP!
             ELSE A3 is abnormal. STOP!
Step 8: 8.1: WEIGHT B1 and B2. GOTO Step 8.2.
        8.2: IF B1 > B2, B1 is abnormal. STOP! ELSE GOTO Step 8.3.
        8.3: IF B1 < B2, B2 is abnormal. STOP!
             ELSE B3 is abnormal. STOP!
Step 9: 9.1: WEIGHT C1 and C2. GOTO Step 9.2.
        9.2: IF C1 > C2, C1 is abnormal. STOP! ELSE GOTO Step 9.3.
        9.3: IF C1 < C2, C2 is abnormal. STOP!
             ELSE C3 is abnormal. STOP!
---------------------------------------------------------------------------
```

For the above algorithm, the number of WEIGHT for finding the abnormal ball is 3.

2 Nine Balls Two Abnormal

Similar to the previous question, you are given a set of 9 balls which are looked and sensed the same. Now, there are two abnormal balls inside and their weights are unknown. Describe in detail, step by step, how do you use the pan balance to find the abnormal balls.

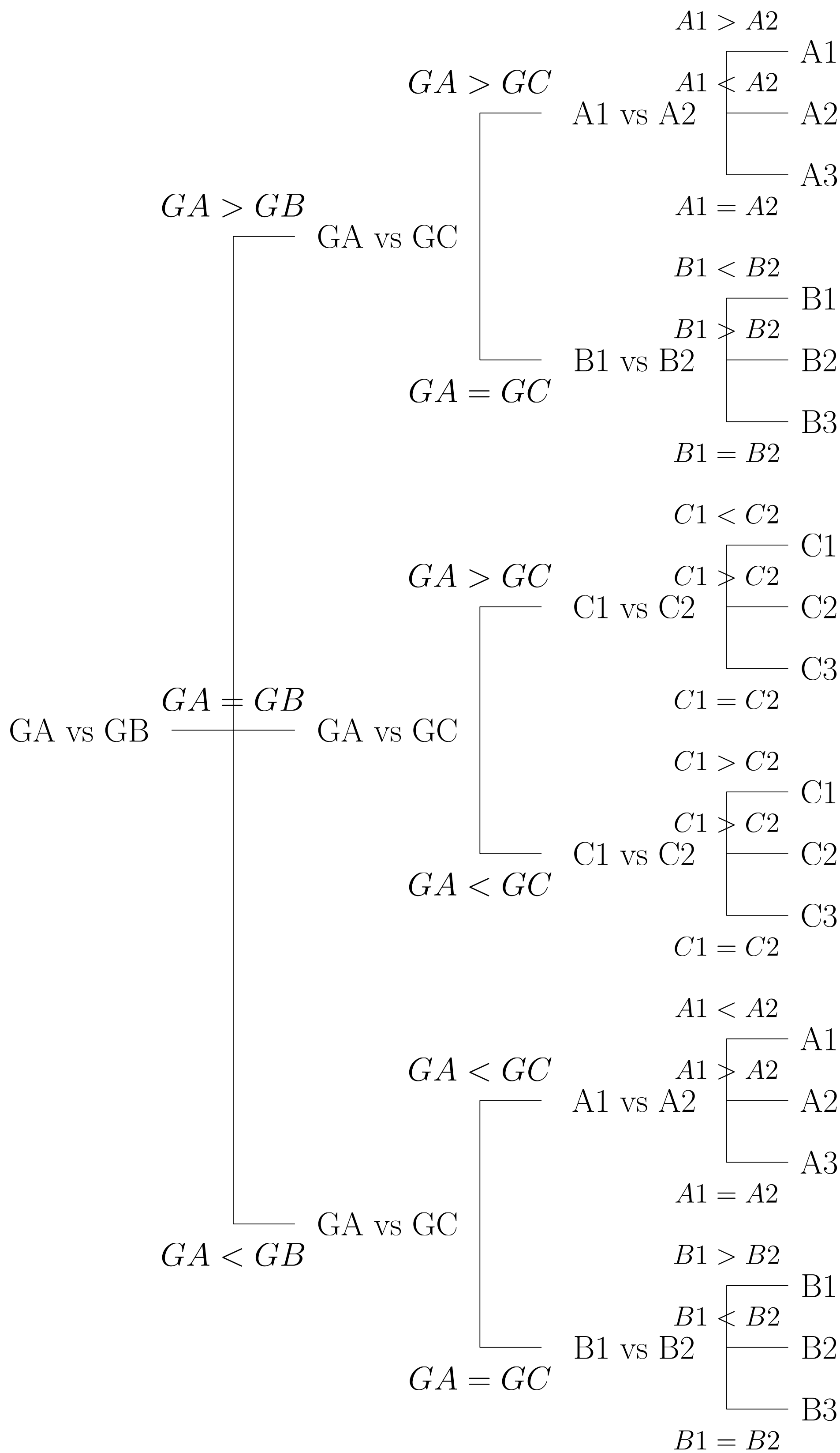

Figure 1: The corresponding decision tree of the *Solution 2* for the nine balls problem, in which the weight of the abnormal ball is unknown.

Answer: Weight B_1 to B_N for $N = 2, \cdots, 9$ and record all the results. Three cases of results will be observed. Let me denote these cases as Case 1.1, Case 1.2 and Case 1.3.

Case 1.1: B_1 is a normal ball. There will be exactly six "=" signs recorded.

B_1 is a normal ball.

(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)	(1, 9)	Abnormal balls
$=$	$=$	$\neq$	$=$	$=$	$=$	$=$	$\neq$	B_3, B_9

Once the above result has been observed, the abnormal balls are identified. There is no further work to be done.

Case 1.2: B_1 is lighter than the normal balls. Thus, the other abnormal ball must be the same weight as B_1 or heavier than B_1. Under such circumstance, there will be at least seven "$\neq$" signs recorded.

B_1 is lighter than the normal balls.

(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)	(1, 9)	Abnormal balls
$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	B_1
$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$=$	B_1, B_9

For the first situation, only B_1 can be identified as an abnormal ball. A second round of weighting is needed. For the second situation, the abnormal balls are identified. There is no further work to be done.

Case 1.3: B_1 is heavier than the normal balls. Thus, the other abnormal ball must be the same weight as B_1 or lighter than B_1. Under such circumstance, there will be at least seven "$\neq$" signs recorded.

B_1 is lighter than the normal balls.

(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)	(1, 9)	Abnormal balls
$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	B_1
$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$=$	B_1, B_9

For the first situation, only B_1 can be identified as an abnormal ball. A second round of weighting is needed. For the second situation, the abnormal balls are identified. There is no further work to be done.

Second Round: As long as B_1 has been identified as an abnormal ball, we could thus weight B_2 to B_N for $N = 3, \cdots, 9$ and record all the results. Again, three cases will be observed. Let me denote these cases as Case 2.1, Case 2.2 and Case 2.3.

Case 2.1: B_2 is a normal ball. There will be exactly five "=" signs recorded.

B_2 is a normal ball.

(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)	(2, 9)	Abnormal balls
$=$	$=$	$=$	$=$	$=$	$=$	$\neq$	B_1, B_9

All abnormal balls are identified. There is no further work to be done.

Case 2.2: B_2 is lighter than the normal balls. There will be exactly seven "$\neq$" signs recorded.

B_2 is lighter than the normal balls.

(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)	(2, 9)	Abnormal balls
$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	B_1, B_2

All abnormal balls are identified. There is no further work to be done.

Case 2.3: B_2 is heavier than the normal balls. There will be exactly at seven "$\neq$" signs recorded.

B_2 is heavier than the normal balls.

$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$	$(2,7)$	$(2,8)$	$(2,9)$	Abnormal balls
$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	$\neq$	B_1, B_2

All abnormal balls are identified. There is no further work to be done.

While the tricks behind the solution for solving the problem are complicated, the procedure for solving the above problem is simple. In summary, the algorithm for solving this 2-abnormal ball problem is depicted as below.

```
---------------------------------------------------------------------------
Step 1: 1.1: FOR N = 2 to 9
               Weight B1 with BN.
               Record the result.
             END
        1.2: IF exactly six '=' signs are recorded,
               the balls with unequal signs are abnormal. STOP!
             ELSE
               B1 is an abnormal ball. GOTO Step 2.1.
             END
Step 2: 2.1: FOR N = 3 to 9
               Weight B2 with BN.
               Record the result.
             END
        2.2: IF exactly six '=' signs are recorded,
                the ball with unequal sign is abnormal. STOP!
             ELSE
                B2 is an abnormal ball. STOP!
             END
---------------------------------------------------------------------------
```

It should be noted that B_1 has already been identified as an abnormal ball if Step 2.1 has to be conducted. Therefore, there is only one abnormal ball in B_2, B_3 to B_9.

3 N Balls M Abnormal

The idea behind the above algorithm can be applied in solving a larger scale problem, in which there are M abnormal balls in a group of N balls. Here $N > 2M$. For instance, one problem is to identify all 10 abnormal balls in a group of 21 balls. In such case, there will require at most 10 rounds of weighting to identify all the abnormal balls.